

# Effects of external fields on two-dimensional Klein-Gordon particle under pseudoharmonic oscillator interaction

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## Abstract

We study the effects of the perpendicular magnetic and Aharonov-Bohm (AB) flux fields on the energy levels of a two-dimensional (2D) Klein-Gordon (KG) particle subjects to equal scalar and vector pseudo-harmonic oscillator (PHO). We calculate the exact energy eigenvalues and normalized wave functions in terms of chemical potential parameter, magnetic field strength, AB flux field and magnetic quantum number by means of the Nikiforov-Uvarov (NU) method. The non-relativistic limit, PHO and harmonic oscillator solutions in the existence and absence of external fields are also obtained.

Keywords: KG equation, two-dimensional PHO potential, HO potential, magnetic field, AB flux field, NU method, bound states.

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## I. INTRODUCTION

The pseudo-harmonic oscillator (PHO) potential is extensively used to describe the bound state of the interaction systems and has been applied for both classical and modern physics. It plays a basic role in chemical and molecular physics since it can be used to calculate the molecular vibration-rotation energy spectrum of linear and non-linear systems [1-3]. This potential is considered as an intermediate between harmonic oscillator (HO) and Morse-type potentials which are more realistic anharmonic potentials. In the non-relativistic quantum mechanics, the PHO is one of the exactly solvable potentials in the frame of the Schrödinger equation and has also been studied in one-dimensional (1D), two-dimensional (2D), three-dimensional (3D) and even in D-dimensional space.

Therefore, this problem has attracted a great deal of interests in solving the Schrödinger and Klein-Gordon (KG) equations with the PHO interaction. The discrete (bound) and continuous (scattering) energy spectra of the PHO have been investigated by the  $SU(1,1)$  spectrum generating algebra [4]. The PHO potential has been obtained depending on the dimension and the angular momentum and the zero-point energy in two-dimension was found to be minimum. The exact polynomial solution of the Schrödinger equation for PHO has been obtained in 3D space [5]. Recently, solutions of the Schrödinger and KG equations with PHO potential have been investigated [6-10]. A realization of the creation and annihilation (ladder) operators for the solution to the Schrödinger equation with a PHO in 2D was studied in Refs. [11,12]. The operators satisfy the commutation relations of an  $SU(1,1)$  group. The exact solution of the Schrödinger equation with a PHO in an arbitrary dimension D was presented in Ref. [13]. The exact analytical solutions of the Schrödinger equation of the D-dimensional space for the PHO potential have been presented by means of the ansatz method and the energy eigenvalues were calculated from the eigenfunction ansatz [14]. The exact bound-state solutions of the KG and the Dirac equations with equal scalar and vector PHO potential have been obtained using the supersymmetric quantum mechanics, shape invariance and other alternative methods [15]. The bound-state solutions of the Dirac equation with PHO have been obtained in the presence of spin and pseudospin symmetries [16]. In relativistic quantum mechanics, the treatment of this problem is relatively inadequate.

Recently, several works have been achieved in this direction for which the solutions of non-relativistic and relativistic equations with different potentials for their bound and continuum

states were investigated. The approximate analytical solutions of the scattering states were derived for  $D$ -dimensional Schrödinger equation with modified Pöschl-Teller, Eckart, modified Morse and generalized Hulthén potentials were found for any angular momentum states using new approximation scheme to centrifugal term [17-20]. It was shown that the energy levels of the continuum states of modified Morse potential reduce to those of bound states at the poles of the scattering amplitude [19]. Using the supersymmetry and shape-invariance, exact bound state solutions of Schrödinger equation with Pöschl-Teller double-ring-shaped Coulomb potential were presented [21]. Exact solution of the one-dimensional KG equation with mixed scalar and vector linear potentials were studied in the context of deformed quantum mechanics characterized by a finite minimal uncertainty in position using the momentum space representation [22]. New ring-shaped harmonic oscillator for spin-1/2 particles was studied and corresponding eigenfunctions and eigenvalues were obtained by using the Dirac equation with equal mixture of scalar and vector potentials [23]. Further, the approximate analytical solution of the Dirac equation with the Pöschl-Teller potential has been presented for arbitrary spin-orbit quantum number in view of spin-symmetry [24].

Recently, the spectral properties in a 2D charged particle (electron or hole) confined by a PHO potential in the presence of external strong uniform magnetic field  $\vec{B}$  along the  $z$  direction and Aharonov-Bohm (AB) flux field created by a solenoid have been studied. The Schrödinger equation is solved exactly for its bound states (energy spectrum and wave functions) [25,26]. So, it is natural that the relativistic effects for a charged particle under the action of this potential could become important, especially for a strong coupling.

The aim of the present work is to study the exact analytical bound state energy eigenvalues and normalized wave functions of the spinless relativistic equation with equal scalar and vector PHO interaction under the effect of external uniform magnetic field and AB flux field in the framework of the NU method [27,28]. The non-relativistic limit of our solution is obtained by making an appropriate mapping of parameters. Further, the KG-PHO and KG-HO special cases are also treated.

The structure of this paper is as follows. We study the effect of external uniform magnetic and AB flux fields on a relativistic spinless particle (anti-particle) under equal scalar and vector PHO interaction in Section 2. We discuss some special cases in Section 3. Finally, we give our concluding remarks in Section 4.

## II. KG EQUATION WITH PHO INTERACTION UNDER EXTERNAL FIELDS

The KG equation is a wave equation mostly used in describing particle dynamics in relativistic quantum mechanics. Nonetheless, physically this equation describes a scalar particle (spin 0). Moreover, this wave equation, for free particles, is constructed using two objects: the four-vector linear momentum operator  $P_\mu = i\hbar\partial_\mu$  and the scalar rest mass  $M$ , allows one to introduce naturally two types of potential couplings. One is the gauge-invariant coupling to the four-vector potential  $\{A_\mu(\vec{r})\}_{\mu=0}^3$  which is introduced via the minimal substitution  $P_\mu \rightarrow P_\mu - gA_\mu$ , where  $g$  is a real coupling parameter. The other, is an additional coupling to the space-time scalar potential  $S_{\text{conf}}(\vec{r})$  which is introduced by the substitution  $M \rightarrow M + S_{\text{conf}}(\vec{r})$ . The term “four-vector” and “scalar” refers to the corresponding unitary irreducible representation of the Poincaré space-time symmetry group (the group of rotations and translations in (3+1)-dimensional Minkowski space-time). Gauge invariance of the vector coupling allows for the freedom to fix the gauge (eliminating the non physical gauge modes) without altering the physical content of the problem. Many choose to simplify the solution of the problem by taking the space component of the vector potential to vanish (i.e.,  $\vec{A}$ ). One may write the time-component of the four-vector potential as  $gA_0 = V_{\text{conf}}(\vec{r})$ , then it ends up with two independent potential functions in the KG equation. These are the “vector”  $V_{\text{conf}}(\vec{r})$  and the “scalar”  $S_{\text{conf}}(\vec{r})$  potentials [29,30].

The free KG equation is written as

$$(\partial^\mu\partial_\mu + M^2c^4)\psi_{KG}(t, \vec{r}) = 0. \quad (1)$$

Moreover, the vector and scalar couplings mentioned above introduce potential interactions by mapping the free KG equation for a 2D single charged electron as

$$\left[ c^2 \left( \vec{p} + \frac{e}{c} \vec{A} \right)^2 - (E - V_{\text{conf}}(\vec{r}))^2 + (Mc^2 + S_{\text{conf}}(\vec{r}))^2 \right] \psi(\vec{r}, \phi) = 0, \quad (2)$$

where we used the transformation  $\vec{p} \rightarrow \vec{p} + \frac{e}{c} \vec{A}$ . Further, the 2D cylindrical wave function  $\psi(\vec{r}, \phi)$  is defined as

$$\psi(\vec{r}, \phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi} g(r), \quad m = 0, \pm 1, \pm 2, \dots, \quad (3)$$

where  $m$  is the magnetic quantum number. This type of coupling attracted a lot of attention in the literature due to the resulting simplification in the solution of the relativistic problem.

The scalar-like potential coupling is added to the scalar mass so that in case when  $S_{\text{conf}}(\vec{r}) = \pm V_{\text{conf}}(\vec{r})$ , the KG equation could always be reduced to a Schrödinger-type second order differential equation as follows

$$\left[ c^2 \left( \vec{p} + \frac{e}{c} \vec{A} \right)^2 + 2 (E \pm M c^2) V_{\text{conf}}(\vec{r}) + M^2 c^4 - E^2 \right] \psi(\vec{r}, \phi) = 0. \quad (4)$$

The potential  $V_{\text{conf}}(\vec{r})$  is taken as the repulsive PHO potential [1-3,11]:

$$V_{\text{conf}}(\vec{r}) = V_0 \left( \frac{r}{r_0} - \frac{r_0}{r} \right)^2, \quad (5)$$

where  $r_0$  and  $V_0$  are the zero point (effective radius) and the chemical potential, respectively. Further, the vector potential  $\vec{A}$  may be represented as a sum of two terms,  $\vec{A} = \vec{A}_1 + \vec{A}_2$  such that  $\vec{\nabla} \times \vec{A}_1 = \vec{B}$  and  $\vec{\nabla} \times \vec{A}_2 = 0$ , where  $\vec{B} = B\hat{z}$  is the applied magnetic field, and  $\vec{A}_2$  describes the additional Aharonov-Bohm (AB) flux field  $\Phi_{AB}$  created by a solenoid in cylindrical coordinates [31]. Hence, the vector potentials have the following azimuthal components [32]

$$\vec{A}_1 = \frac{Br}{2} \hat{\phi}, \quad \vec{A}_2 = \frac{\Phi_{AB}}{2\pi r} \hat{\phi}, \quad \vec{A} = \left( \frac{Br}{2} + \frac{\Phi_{AB}}{2\pi r} \right) \hat{\phi}. \quad (6)$$

Hence, the bound-state solutions of the two cases in Eq. (4) are to be treated separately as follows.

### A. The bound states for positive energy

The positive energy states (corresponding to  $S_{\text{conf}}(\vec{r}) = +V_{\text{conf}}(\vec{r})$  in the non-relativistic limit (taking  $E - M c^2 \rightarrow E$  and  $(E + M c^2) \rightarrow 2\mu c^2$ , where  $M$  is an effective mass and  $|E| \ll M c^2$ ) are solutions of

$$\left[ \frac{1}{2\mu} \left( \vec{p} + \frac{e}{c} \vec{A} \right)^2 + 2V_{\text{conf}}(\vec{r}) - E \right] \psi(\vec{r}, \phi) = 0, \quad (7)$$

where  $\psi(\vec{r}, \phi)$  stands for either  $\psi^{(+)}(\vec{r}, \phi)$  or  $\psi^{(\text{KG})}(\vec{r}, \phi)$ . This is the Schrödinger equation for the potential  $2V_{\text{conf}}(\vec{r})$ . Thus, the choice  $S_{\text{conf}}(\vec{r}) = +V_{\text{conf}}(\vec{r})$  produces a nontrivial non-relativistic limit with a potential function  $2V_{\text{conf}}(\vec{r})$ , and not  $V_{\text{conf}}(\vec{r})$ . Accordingly, it would be natural to scale the potential term in Eq. (4) and Eq. (7) so that in the non-relativistic limit the interaction potential becomes  $V_{\text{conf}}$ , not  $2V_{\text{conf}}$ . Thus, we need to recast Eq. (4) for the  $S(\vec{r}) = V(\vec{r})$  as [29,33]

$$\left[ c^2 \left( -i\hbar \vec{\nabla} + \frac{e}{c} \vec{A} \right)^2 + 2 (E + M c^2) V_{\text{conf}}(\vec{r}) \right] \psi(\vec{r}, \phi) = (E^2 - M^2 c^4) \psi(\vec{r}, \phi), \quad (8)$$

and in order to simplify Eq. (8) we introduce new parameters  $\lambda_1 = E + Mc^2$  and  $\lambda_2 = E - Mc^2$  so that it can be reduced to the form

$$\left[ c^2 \left( -i\hbar \vec{\nabla} + \frac{e}{c} \vec{A} \right)^2 - \lambda_1 (\lambda_2 - V_{\text{conf}}(\vec{r})) \right] \psi(\vec{r}, \phi) = 0. \quad (9)$$

Now, inserting Eqs. (3), (5) and (6) into the KG equation (9) and further introducing a change of variable  $s = r^2$ , that maps  $r \in (0, \infty)$  to  $s \in (0, \infty)$ , we obtain second-order differential equation satisfying the radial wave function  $g(s)$ ,

$$g''(s) + \frac{1}{s} g'(s) + \frac{1}{s^2} \left( -\frac{\gamma^2}{4} s^2 + \frac{\nu^2}{4} s - \frac{\beta^2}{4} \right) g(s) = 0, \quad g(0) = 0 \text{ and } g(\infty) = 0, \quad (10)$$

where we have employed the following abbreviation symbols

$$\nu^2 = \frac{1}{\hbar^2 c^2} [\lambda_1 (\lambda_2 + 2V_0) - Mc^2 \omega_c \hbar m'], \quad (11a)$$

$$\beta^2 = m'^2 + \frac{1}{\hbar^2 c^2} r_0^2 V_0 \lambda_1, \quad m' = m + \xi, \quad m' = 1, 2, \dots, \quad (11b)$$

$$\gamma^2 = \left( \frac{M\omega_c}{2\hbar} \right)^2 + \frac{1}{\hbar^2 c^2} \frac{V_0 \lambda_1}{r_0^2}, \quad (11c)$$

with  $m'$  is a new quantum number. Here  $\xi = \Phi_{AB}/\Phi_0$  is taken as integer with the flux quantum  $\Phi_0 = hc/e$ ,  $\omega_c = eB/Mc$  is the cyclotron frequency. Now, we use the basic ideas of the NU method [27] and the parametric NU derived in Ref. [28] to obtain energy spectrum equation

$$\nu^2 = 2(2n + 1 + \beta) \gamma, \quad n = 0, 1, 2, \dots \quad (12)$$

where the constant parameters used in our calculations are displayed in Table 1. Inserting Eqs. (11a)-(11c), we finally arrive at the following transcendental energy formula,

$$\begin{aligned} & 2 \left( 1 + 2n + \sqrt{m'^2 + \frac{V_0 r_0^2 \lambda_1}{\hbar^2 c^2}} \right) \sqrt{\left( \frac{M\omega_c}{2\hbar} \right)^2 + \frac{V_0 \lambda_1}{\hbar^2 c^2 r_0^2}} \\ &= \frac{1}{\hbar^2 c^2} [\lambda_1 (\lambda_2 + 2V_0) - Mc^2 \hbar \omega_c m'], \quad m' = 1, 2, \dots, \end{aligned} \quad (13)$$

We may find solution to the above transcendental equation as  $E = E_{KG}^{(+)}$ . In the non-relativistic limit when  $\lambda_1 \rightarrow 2Mc^2$  and  $\lambda_2 \rightarrow E$ , the above energy equation has a solution given by Eq. (39) of Ref. [25].

Using Eq. (38) of Ref. [28] and Table 1, we find the corresponding radial wave function  $g(r)$  as

$$g(r) = C_{n,m} r^{|\beta|} e^{-\gamma r^2/2} F(-n, |\beta| + 1; \gamma r^2), \quad (14)$$

and hence the total 2D KG wave function (3) takes the explicit form

$$\psi_{n,m}^{(+)}(\vec{r}, \phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi} \sqrt{\frac{\gamma^{|\beta|+1} n!}{\pi (n + |\beta|)!}} r^{|\beta|} e^{-\gamma r^2/2} L_n^{(\beta)}(\gamma r^2), \quad (15)$$

where  $L_a^{(b)}(x) = \frac{(a+b)!}{a!b!} F(-a, b+1; x)$  is the associated Laguerre polynomial and  $F(-a, b; x)$  is the confluent hypergeometric function. Note that the wave function (15) is finite and satisfying the standard asymptotic analysis for  $r = 0$  and  $r \rightarrow \infty$ .

## B. The bound states for negative energy

When  $S_{\text{conf}}(\vec{r}) = -V_{\text{conf}}(\vec{r})$ , we need to follow same procedure of solution in subsection A and consider the solution given by (12) with the changes

$$\nu^2 \rightarrow \tilde{\nu}^2 = \frac{1}{\hbar^2 c^2} [\lambda_2 (\lambda_1 + 2V_0) - Mc^2 \omega_c \hbar m'], \quad (16a)$$

$$\beta^2 \rightarrow \tilde{\beta}^2 = m'^2 + \frac{1}{\hbar^2 c^2} r_0^2 V_0 \lambda_2, \quad (16b)$$

$$\gamma^2 \rightarrow \tilde{\gamma}^2 = \left( \frac{M\omega_c}{2\hbar} \right)^2 + \frac{1}{\hbar^2 c^2} \frac{V_0 \lambda_2}{r_0^2}. \quad (16c)$$

Hence, the negative energy solution for antiparticle can be readily found as

$$\begin{aligned} & 2 \left( 2n + 1 + \sqrt{m'^2 + \frac{1}{\hbar^2 c^2} r_0^2 V_0 \lambda_2} \right) \sqrt{\left( \frac{M\omega_c}{2\hbar} \right)^2 + \frac{1}{\hbar^2 c^2} \frac{V_0 \lambda_2}{r_0^2}} \\ &= \frac{1}{\hbar^2 c^2} [\lambda_2 (\lambda_1 + 2V_0) - Mc^2 \hbar \omega_c m'], \quad m' = 1, 2, \dots, \end{aligned} \quad (17)$$

and the 2D wave function is

$$\psi_{n,m}^{(-)}(\vec{r}, \phi) = \sqrt{\frac{\tilde{\gamma}^{|\tilde{\beta}|+1} n!}{\pi (n + |\tilde{\beta}|)!}} r^{|\tilde{\beta}|} e^{-\tilde{\gamma} r^2/2} L_n^{(\tilde{\beta})}(\tilde{\gamma} r^2) \frac{1}{\sqrt{2\pi}} e^{im\phi}. \quad (18)$$

It should be noted that the negative energy states are free fields since under these conditions Eq. (4) can be rewritten as

$$\left[ -\frac{1}{2\mu} \left( \vec{p} + \frac{e}{c} \vec{A} \right)^2 + E \right] \psi_{n,m}(\vec{r}, \phi) = 0, \quad (19)$$

which is a simple free-interaction mode. Further, the set of parameters given in Eqs. (16a)-(16c) becomes

$$\tilde{\nu} = \sqrt{\frac{2ME}{\hbar^2} - \frac{M\omega_c}{\hbar} m'}, \quad \tilde{\beta} = m', \quad \tilde{\gamma} = \frac{M\omega_c}{2\hbar}. \quad (20)$$

Thus, Eq. (17) gives the following energy formula

$$E_{nm'}^{(-)} = \left( n + m' + \frac{1}{2} \right) \hbar \omega_c, \quad (21)$$

and hence the wave function reads

$$\psi_{nm'}^{(-)}(\vec{r}, \phi) = \sqrt{\frac{\left(\frac{M\omega_c}{2\hbar}\right)^{m'+1} n!}{\pi (n+m')!}} r^{m'} e^{-\frac{M\omega_c}{4\hbar c} r^2} L_n^{(m')} \left( \frac{M\omega_c}{2\hbar} r^2 \right) e^{im\phi}. \quad (22)$$

### III. DISCUSSIONS

In this section we briefly study some special cases and relationship between our results and some other authors':

#### A. Schrödinger-PHO problem under the effect of magnetic and AB flux fields

In the non-relativistic limit ( $\lambda_1 \rightarrow 2M$  and  $\lambda_2 \rightarrow E$ ), Eq. (13) can be easily reduced into the form:

$$E_{nm}(\xi, \beta) = \hbar \Omega \left( n + \frac{|\tilde{m}| + 1}{2} \right) + \frac{1}{2} \hbar \omega_c m' - 2V_0, \quad (23a)$$

$$\Omega = \sqrt{\omega_c^2 + 4\omega_D^2}, \quad \omega_D = \sqrt{2V_0/Mr_0^2}, \quad |\tilde{m}| = \sqrt{m'^2 + a^2}, \quad (23b)$$

where  $a = k_F r_0$  with  $k_F = \sqrt{2MV_0/\hbar^2}$  is the fermi wave vector of the electron [25]. Further, the wave function becomes

$$\psi_{n,m}^{(+)}(\vec{r}, \phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi} \sqrt{\frac{c^{b+1} n!}{\pi (n+b)!}} r^b e^{-cr^2/2} L_n^{(b)}(cr^2), \quad (24)$$

with the given parameters

$$b = \sqrt{m'^2 + \frac{2MV_0 r_0^2}{\hbar^2}} \quad \text{and} \quad c = \sqrt{\left(\frac{M\omega_c}{2\hbar}\right)^2 + \frac{2MV_0}{\hbar^2 r_0^2}}.$$

#### B. KG-PHO problem

The energy spectrum of relativistic spinless particle in the absence of magnetic and AB flux fields has the form

$$\frac{2\hbar}{r_0} \sqrt{V_0} \left( 1 + 2n + \sqrt{m^2 + \frac{V_0 r_0^2}{\hbar^2} \lambda_1} \right) = (\lambda_2 + 2V_0) \sqrt{\lambda_1}, \quad (25)$$



and is reduced to its non-relativistic limit:

$$E_{nm} = -2V_0 + \left(1 + 2n + \sqrt{m^2 + \frac{2MV_0r_0^2}{\hbar^2}}\right) \sqrt{\frac{2V_0\hbar^2}{Mr_0^2}}, \quad (26)$$

which is completely identical to Eq. (27) of Ref. [7]. The wave function can be expressed as

$$\psi_{n,m}^{(+)}(\vec{r}, \phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi} \sqrt{\frac{C^{B+1}n!}{\pi(n+B)!}} r^B e^{-Cr^2/2} L_n^{(B)}(Cr^2), \quad (27)$$

with

$$B = \sqrt{m^2 + \frac{r_0^2 V_0}{\hbar^2}} \lambda_1 \quad \text{and} \quad C = \frac{1}{\hbar r_0} \sqrt{V_0 \lambda_1}.$$

### C. KG-HO problem

The energy equation of the relativistic spinless particle subject to the HO field is

$$n' \hbar c \sqrt{2k} - \sqrt{\lambda_1} \lambda_2 = 0, \quad n' = 1, 2, \dots, \quad (28)$$

where  $n' = 1 + |m| + 2n$ ,  $n = 0, 1, 2, \dots$ , which is completely identical to Eq. (11) and Eq. (26) of Ref. [34] when one uses the notation  $k = 2V_0/r_0^2 = M\omega_D'^2$ . Following Ref. [35], the Eq. (28) has three solutions, the only real solution giving energy is

$$E_{nm} = \frac{1}{3} (Mc^2 + M^2 c^4 T^{-1/3} + T^{1/3}), \quad (29)$$

with

$$T = 27kn'^2 \hbar^2 c^2 - 8M^3 c^6 + 3n' \hbar c \sqrt{3k(27kn'^2 \hbar^2 c^2 - 16M^3 c^6)}. \quad (30)$$

The wave function takes the form

$$\psi_{n,m}^{(+)}(\vec{r}, \phi) = \sqrt{\frac{D^{|m|+1}n!}{\pi(n+|m|)!}} r^{|m|} e^{-Dr^2/2} L_n^{(|m|)}(Dr^2) e^{im\phi}, \quad D = \frac{1}{\hbar r_0} \sqrt{V_0 \lambda_1}. \quad (31)$$

If one expands Eq. (28) as a series of  $\lambda_2$ , it becomes

$$n' \hbar = \sqrt{\frac{M}{k}} \left[ \lambda_2 + \frac{1}{4Mc^2} \lambda_2^2 - \frac{1}{32M^2 c^4} \lambda_2^3 + O(\lambda_2^4) \right], \quad (32)$$

and taking the first order of  $\lambda_2$  by neglecting the higher order relativistic corrections, we finally arrive at the non-relativistic solution:

$$E'_{nm} = E_{nm} - Mc^2 = \hbar \sqrt{\frac{k}{M}} (1 + 2n + |m|) = (1 + |m| + 2n) \hbar \omega'_D, \quad n = 0, 1, 2, \dots, \quad (33)$$

and wave function resembles the one given in (31) with  $D = \frac{1}{\hbar r_0} \sqrt{2MV_0}$ .

#### IV. CONCLUDING REMARKS

In this work, we have obtained bound state energies and wave functions of the relativistic spinless particle subject to a PHO interaction and expressed in terms of external uniform magnetic field and AB flux field. We explored the solution of both positive (particle) and negative (anti-particle) KG energy states. The Schrödinger bound state solution is found as non-relativistic limit of the present model. It is noticed that the solution with equal mixture of scalar and vector potentials can be easily reduced into the well-known Schrödinger solution for a particle with an interaction potential field and a free field, respectively. We have also studied the bound-state solutions for some special cases including the non-relativistic limits (Schrödinger equation for PHO and HO under external magnetic and AB flux fields) and the KG equation for HO and PHO interactions. The results show that the splitting is not constant and dependent on the strength of the external magnetic field and AB flux field.

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#### Appendix A: A Short Solution to KG Bound States

Here, we seek to find a short solution to Eq. (9). By substituting Eqs. (3), (5) and (6) into Eq. (9), we obtain a  $2D$  Schrödinger-type equation for the anharmonic oscillator  $V(r) = \gamma^2 r^2$  satisfying the radial wave function  $R(r)$ ,

$$\left( \frac{d^2}{dr^2} - \frac{(\beta^2 - 1/4)}{r^2} + \nu^2 - \gamma^2 r^2 \right) R(r) = 0, \quad (\text{A1})$$

where  $g(r) = r^{-1/2}R(r)$ . For brevity, the solution of the above equation can be found through Eq. (17) of Ref. [5] by simply making the changes:  $2\mu E'_{nL}/\hbar^2 \rightarrow \nu^2$ ,  $2\mu B^2/\hbar^2 \rightarrow \gamma^2$  and  $2L + 1 \rightarrow 2\beta$  in Eq. (19) of Ref. [5] and then finally obtain our Eqs. (12) and (13). A first inspection on the asymptotic behavior of Eq. (A1), we find out that when  $r$  approaches 0, the radial wave function  $R(r) \sim r^q$ ,  $q = \beta + 1/2 > 0$  and when  $r \rightarrow \infty$ ,  $R(r) \sim \exp(-\gamma r^2/2)$

and both solutions are satisfying the finiteness of the radial wave function  $g(r=0)=0$  and  $g(r\rightarrow\infty)\rightarrow 0$ . Therefore, for the entire range  $r\in(0,\infty)$ , we consider the general solution  $g(r)=r^\beta\exp(-\gamma r^2/2)h(r)$ ,  $\beta>0$ , where  $h(r)$  is the associated Laguerre polynomials, i.e.,  $h(r)=L_n^{(a)}(br^2)$ . With these behaviors, the 3D wave function (20) of Ref. [5] gives the 2D wave function as in Eq. (15) when one uses the relation:

$$\begin{aligned} g(r) &= Nr^{-1/2}R(r) = Nr^{L+1/2}\exp(-\sqrt{\frac{\mu}{2\hbar^2}}Br^2)L_n^{(L+1/2)}\left(\sqrt{\frac{2\mu}{\hbar^2}}Br^2\right) \\ &= Nr^\beta\exp(-\frac{1}{2}\gamma r^2)L_n^{(\beta)}(\gamma r^2), \end{aligned} \quad (\text{A2})$$

where  $N$  is the normalization constant.

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TABLE I: Specific values of the constants in the solution of Eq. (20).

| Constants:                                      | ( $\alpha_3 = 0$ case)          |
|---|---------------------------------|
| $\xi_1 = \gamma^2/4$                            | $\xi_2 = \nu^2/4$               |
| $\xi_3 = \beta^2/4$                             | $\alpha_1 = 1$                  |
| $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0$ | $\alpha_6 = \xi_1 = \gamma^2/4$ |
| $\alpha_7 = -\xi_2 = -\nu^2/4$                  | $\alpha_8 = \xi_3 = \beta^2/4$  |
| $\alpha_9 = \alpha_6 = \gamma^2/4$              | $\alpha_{10} = \beta + 1$       |
| $\alpha_{11} = \gamma$                          | $\alpha_{12} = \beta/2$         |
| $\alpha_{13} = -\gamma/2$                       |                                 |